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Phase of Bose-Einstein Condensate Interacting with a Time-Dependent Laser Field

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By using of the invariant theory, we have studied phase of Bose-Einstein condensate in a double-well potential interacting with a time-dependent single-mode traveling-wave laser field, the dynamical and geometric phases are presented respectively. The Aharonov-Anandan phase is also obtained under the cyclical evolution.

KEY WORDS: phase; Bose-Einstein condensation.

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1. INTRODUCTION

Recently, much attention has been paid to the investigation of Bose-Einstein condensation (BEC) in dilute and ultracold gases of neutral alkali-metal atoms using a combination of laser and evaporative cooling (Anderson *et al.*, 1995; Bradley *et al.*, 1995; Davis *et al.*, 1995; Mewes *et al.*, 1996a,b, 1997; Jin *et al.*, 1996) due to the optical properties (Politzer *et al.*, 1991, 1997; Lewenstein *et al.*, 1994, 1995; Lewenstein *et al.*, 1993, 1994; Javanainen *et al.*, 1994, 1995a,b, 1996), statistical properties (Grossman *et al.*, 1995, 1994, 1996, 1997a,b; Kuang *et al.*, 1998, 2001), phase properties (Javanainen *et al.*, 1994, 1995a,b 1996; Javanainen *et al.*, 1996, 1997; Javanainen *et al.*, 1995, 1997a,b; Imamoglu *et al.*, 1997; Wong *et al.*, 1996a,b; Cirac *et al.*, 1996, 2002, 2003, 2001; Castin *et al.*, 1997), and tunneling effect (Javanainen *et al.*, 1986, 1991; Jack *et al.*, 1997; Milburn *et al.*, 1997; Grossman *et al.*, 1995; Kuang *et al.*, 2000, 2001; Wu *et al.*, 2000; Wu *et al.*,

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1996; Wu *et al.*, 2001, 2006; Liu *et al.*, 2002; Liu *et al.*, 2000; Niu *et al.*, 1999; Liang *et al.*, 2003; Li *et al.*, 2001).

As we known that the quantum invariant theory proposed by Lewis and Riesenfeld (Lewis *et al.*, 1969) is a powerful tool for treating systems with time-dependent Hamiltonians. It was generalized in (Gao *et al.*, 1991) by introducing the concept of basic invariants and used to study the geometric phases (Berry *et al.*, 1984; Aharonov *et al.*, 1987) in connection with the exact solutions of the corresponding time-dependent Schrödinger equations. The discovery of Berry's phase is not only a breakthrough in the older theory of quantum adiabatic approximations (Berry, 1984; Simon, 1983), but also provides us with new insights in many physical phenomena. The concept of Berry's phase has developed in some different directions (Richardson *et al.*, 1988; Wilczek *et al.*, 1984; Moody *et al.*, 1986, 1987; Sun *et al.*, 1990; Sun, 1998 a,b; Sun, 1998; Sun, 2001).

Recently, the dynamics of an atomic Bose-Einstein condensate in a double-well potential interacting with a single-mode traveling-wave laser field has been studied under electric dipole and rotating-wave approximations (Wang *et al.*, 2000). In this paper, by using of the invariant theory, we shall study the dynamical and the geometric phases of Bose-Einstein condensate in a double-well potential interacting with a time-dependent single-mode traveling-wave laser field.

2. MODEL

We consider that the atoms are trapped in a symmetrical potential denoted by $V(x) = \frac{1}{2}M\omega^2(|x| - d)^2$, where M is the atomic mass, ω the angular frequency, d half the distance between the two minima of the potential $V(x)$. This potential has two wells and hereafter they will be called the left (L) and right (R) well. When a time-dependent single-mode traveling-wave laser field is applied, ignoring the noncondensed atoms and letting $E_0 = 0$, we can obtain the Hamiltonian of this system according to the Jaynes-Cummings model (Jaynes *et al.*, 1963) and using the same treatment method proposed in (Wang *et al.*, 2000) (in the unit of $\bar{h} = 1$),

$$\hat{H}(t) = (\omega_a + \Omega_0)\hat{c}^\dagger\hat{c} + (\omega_a - \Omega_0)\hat{d}^\dagger\hat{d} + \omega_f\hat{a}^\dagger\hat{a} + g\sqrt{N}_c(\hat{c}^\dagger\hat{a}e^{i\triangle t} + \hat{a}^\dagger\hat{c}e^{-i\triangle t}). \tag{1}$$

The difference between Eq. (1) and the Hamiltonian given in (Wang, 2000) lies in that a time-dependent single-mode traveling-wave laser field is considered. In Eq. (1), ω_a (in the unit of $\bar{h} = 1$) is the energy interval between the ground and the excited states, Ω_0 is the tunneling frequency of the ground state, $\hat{a}^\dagger$ and $\hat{a}$ are the creation and annihilation operators for a photon with energy ω_f (in the unit of $\bar{h} = 1$), $\Delta = \omega_f - \omega_a$ is the detuning. g denotes the dipole coupling constant, N_c is the condensed atomic number in the ground state. $\hat{c}^\dagger$ and $\hat{d}^\dagger$ are the Hermitian adjoint operators of $\hat{c}$ and $\hat{d}$ defined by

$$\hat{c} = \frac{1}{\sqrt{2}}(\hat{b}_{L_0} + \hat{b}_{R_0}), \quad \hat{d} = \frac{1}{\sqrt{2}}(\hat{b}_{L_0} - \hat{b}_{R_0}). \tag{2}$$

3. GEOMETRIC AND DYNAMICAL PHASES

For self-consistent, we first illustrate the Lewis-Riesenfeld (L-R) invariant theory Lewis and Risenteed, 1969. For a one-dimensional system whose Hamiltonian $\hat{H}(t)$ is time-dependent, then there exists an operator $\hat{I}(t)$ called invariant if it satisfies the equation

$$i\frac{\partial \hat{I}(t)}{\partial t} + [\hat{I}(t), \hat{H}(t)] = 0. \tag{3}$$

The eigenvalue equation of the time-dependent invariant $|\lambda_n, t\rangle$ is given

$$\hat{I}(t)|\lambda_n, t\rangle = \lambda_n|\lambda_n, t\rangle, \tag{4}$$

where $\frac{\partial \lambda_n}{\partial t} = 0$. The time-dependent Schrödinger equation for this system is

$$i\frac{\partial|\psi(t)\rangle_s}{\partial t} = \hat{H}(t)|\psi(t)\rangle_s. \tag{5}$$

According to the L-R invariant theory, the particular solution $|\lambda_n, t\rangle_s$ of Eq. (5) is different from the eigenfunction $|\lambda_n, t\rangle$ of $\hat{I}(t)$ only by a phase factor $\exp[i\delta_n(t)]$, i.e.,

$$|\lambda_n, t\rangle_s = \exp[i\delta_n(t)]|\lambda_n, t\rangle, \tag{6}$$

which shows that $|\lambda_n, t\rangle_s$ $(n = 1, 2, \ldots)$ forms a complete set of the solutions of Eq. (5). Then the general solution of the Schrödinger equation (5) can be written by

$$|\psi(t)\rangle_s = \sum_n C_n \exp[i\delta_n(t)]|\lambda_n, t\rangle, \tag{7}$$

where

$$\delta_n(t) = \int_0^t dt' \left\langle \lambda_n, t' \left| i\frac{\partial}{\partial t'} - \hat{H}(t')|\lambda_n, t' \right. \right\rangle, \tag{8}$$

and $C_n = \langle \lambda_n, 0|\psi(0)\rangle_s$.

In order to obtain the exact solution of Eq. (5), we rewrite Eq. (1) as

$$\hat{H} = \hat{H}^{(1)} + \hat{H}^{(2)}, \tag{9}$$

where

$$\hat{H}^{(1)} = (\omega_a - \Omega_0)\hat{d}^{\dagger}\hat{d}, \tag{10}$$

$$\hat{H}^{(2)} = (\omega_a + \Omega_0)\hat{c}^\dagger\hat{c} + \omega_f \hat{a}^\dagger\hat{a} + g\sqrt{N_c}(\hat{c}^\dagger\hat{a}e^{i\triangle t} + \hat{a}^\dagger\hat{c}e^{-i\triangle t}), \tag{11}$$

one has $[\hat{H}^{(1)}, \hat{H}^{(2)}] = 0$. Furthermore, we define operators $\hat{K}_+$, $\hat{K}_-$ and $\hat{K}_0$ as follows:

$$\hat{K}_+ = \hat{a}^\dagger\hat{c}, \quad \hat{K}_- = \hat{c}^\dagger\hat{a}, \quad \hat{K}_0 = \hat{a}^\dagger\hat{a} - \hat{c}^\dagger\hat{c}, \tag{12}$$

which hold the commutation relations

$$[\hat{K}_0, \hat{K}_\pm] = \pm 2\hat{K}_\pm, \quad [\hat{K}_+, \hat{K}_-] = \hat{K}_0, \tag{13}$$

it is easy to prove that operators $\hat{K}_+$, $\hat{K}_-$ and $\hat{K}_0$ together with the Hamiltonian $\hat{H}^{(2)}$ construct a quasi-algebra.

Then we can get the L-R invariant as follows

$$\hat{I}(t) = \cos\theta\hat{K}_0 - e^{-i\varphi}\sin\theta\hat{K}_+ - e^{i\varphi}\sin\theta\hat{K}_-, \tag{14}$$

here θ and φ are determined by $i\partial\hat{I}(t)/\partial t + [\hat{I}(t), \hat{H}^{(2)}(t)] = 0$, and satisfy the relations

$$\dot{\theta} = 2g\sqrt{N_c}\sin(\varphi - \triangle t), \tag{15}$$

$$\begin{aligned} &\dot{\theta}\cos\theta\sin\varphi + (\dot{\varphi} + \omega_a + \Omega_0 - \omega_f)\sin\theta\cos\varphi \\ &\quad - 2g\sqrt{N_c}\cos\theta\cos\triangle t = 0, \end{aligned} \tag{16}$$

$$\begin{aligned} &\dot{\theta}\cos\theta\cos\varphi - (\dot{\varphi} + \omega_a + \Omega_0 - \omega_f)\sin\theta\sin\varphi \\ &\quad + 2g\sqrt{N_c}\cos\theta\sin\triangle t = 0, \end{aligned} \tag{17}$$

where dot denotes the time derivative.

We can construct the unitary transformation

$$\hat{V}(t) = \exp[\sigma\hat{K}_+ - \sigma^*\hat{K}_-], \tag{18}$$

where $\sigma = \frac{\theta}{2}e^{-i\varphi}$ and $\sigma^* = \frac{\theta}{2}e^{i\varphi}$. The invariant $\hat{I}(t)$ can be transformed into a new time-independent operator $\hat{I}_V$:

$$\hat{I}_V = \hat{V}^\dagger(t)\hat{I}(t)\hat{V}(t) = \hat{K}_0. \tag{19}$$

Correspondingly, we can get the eigenvalue equation of operator $\hat{I}_V(t)$

$$\hat{I}_V|m\rangle_{\hat{a}}|n\rangle_{\hat{c}} = (m-n)|m\rangle_{\hat{a}}|n\rangle_{\hat{c}}, \tag{20}$$

where we have used $\hat{a}^\dagger\hat{a}|m\rangle_{\hat{a}} = m|m\rangle_{\hat{a}}$ and $\hat{c}^\dagger\hat{c}|n\rangle_{\hat{c}} = n|n\rangle_{\hat{c}}$.

In terms of the unitary transformation $\hat{V}(t)$ and the Baker-Campbell-Hausdoff formula (Wei *et al.*, 1963)

$$\hat{V}^{\dagger}(t)\frac{\partial \hat{V}(t)}{\partial t} = \frac{\partial \hat{L}}{\partial t} + \frac{1}{2!}\left[\frac{\partial \hat{L}}{\partial t}, \hat{L}\right] + \frac{1}{3!}\left[\left[\frac{\partial \hat{L}}{\partial t}, \hat{L}\right], \hat{L}\right] + \frac{1}{4!}\left[\left[\left[\frac{\partial \hat{L}}{\partial t}, \hat{L}\right], \hat{L}\right], \hat{L}\right] + \ldots, \tag{21}$$

where $\hat{V}(t) = \exp[\hat{L}(t)]$, one has

$$\begin{aligned}\hat{H}_V^{(2)}(t) &= \hat{V}^{\dagger}(t)\hat{H}^{(2)}(t)\hat{V}(t) - i\hat{V}^{\dagger}(t)\frac{\partial \hat{V}(t)}{\partial t} \\ &= \left[(\omega_a + \Omega_0)\sin^2\frac{\theta}{2} + \omega_f\cos^2\frac{\theta}{2} - g\sqrt{N_c}\sin\theta\cos(\varphi - \triangle t)\right. \\ &\quad \left. + \frac{\dot{\varphi}}{2}(1 - \cos\theta)\right]\hat{a}^{\dagger}\hat{a} + \left[(\omega_a + \Omega_0)\cos^2\frac{\theta}{2} + \omega_f\sin^2\frac{\theta}{2}\right. \\ &\quad \left. + g\sqrt{N_c}\sin\theta\cos(\varphi - \triangle t) - \frac{\dot{\varphi}}{2}(1 - \cos\theta)\right]\hat{c}^{\dagger}\hat{c}.\end{aligned} \tag{22}$$

It is easy to find that $\hat{H}^{(2)}(t)$ differs from $\hat{I}_V$ only by a time-dependent c-number factor. Thus we can get the general solution of the time-dependent Schrödinger equation Eq. (5)

$$|\Psi(t)\rangle_s = \sum_m\sum_n\sum_l C_{mn}C_l \exp[i\delta_{mn}(t)]\hat{V}(t)|m>_{\hat{a}}\ |n>_{\hat{c}}\ |l\rangle_{\hat{d}}, \tag{23}$$

with the coefficients $C_{mn} = \langle m, n, t = 0|\Psi(0)\rangle_s$, and $\hat{d}^{\dagger}\hat{d}|l\rangle_{\hat{d}} = l|l\rangle_{\hat{d}}$. The phase $\delta_{mn}(t) = \delta^d_{mn}(t) + \delta^g_{mn}(t)$ includes the dynamical phase

$$\begin{aligned}\delta^d_{mn}(t) = &-m\int_{t_0}^{t}\left[(\omega_a + \Omega_0)\sin^2\frac{\theta}{2} + \omega_f\cos^2\frac{\theta}{2} - g\sqrt{N_c}\sin\theta\cos(\varphi - \triangle t)\right]dt' \\ &-n\int_{t_0}^{t}\left[(\omega_a + \Omega_0)\cos^2\frac{\theta}{2} + \omega_f\sin^2\frac{\theta}{2} + g\sqrt{N_c}\sin\theta\cos(\varphi - \triangle t)\right]dt' \\ &+l\int_{t_0}^{t}(\omega_a - \Omega_0)dt',\end{aligned} \tag{24}$$

and the geometric phase

$$\delta^g_{mn}(t) = (n - m)\int_{t_0}^{t}\frac{\dot{\varphi}}{2}(1 - \cos\theta)dt'. \tag{25}$$

In particular, the geometric phase becomes in the case of the cyclical evolution

$$\delta_{mn}^{g}(t) = \frac{1}{2}(n-m)\oint(1-\cos\theta)d\varphi, \tag{26}$$

which is the geometric Aharonov-Anandan phase.

4. CONCLUSIONS

In conclusion, by using of the L-R invariant theory, we have studied the dynamical and the geometric phases of Bose-Einstein condensate in a double-well potential interacting with a time-dependent single-mode traveling-wave laser field. The dynamical and geometric phases are presented respectively. The Aharonov-Anandan phase is obtained under the cyclical evolution.

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